The Geometry of Interest Rate Risk

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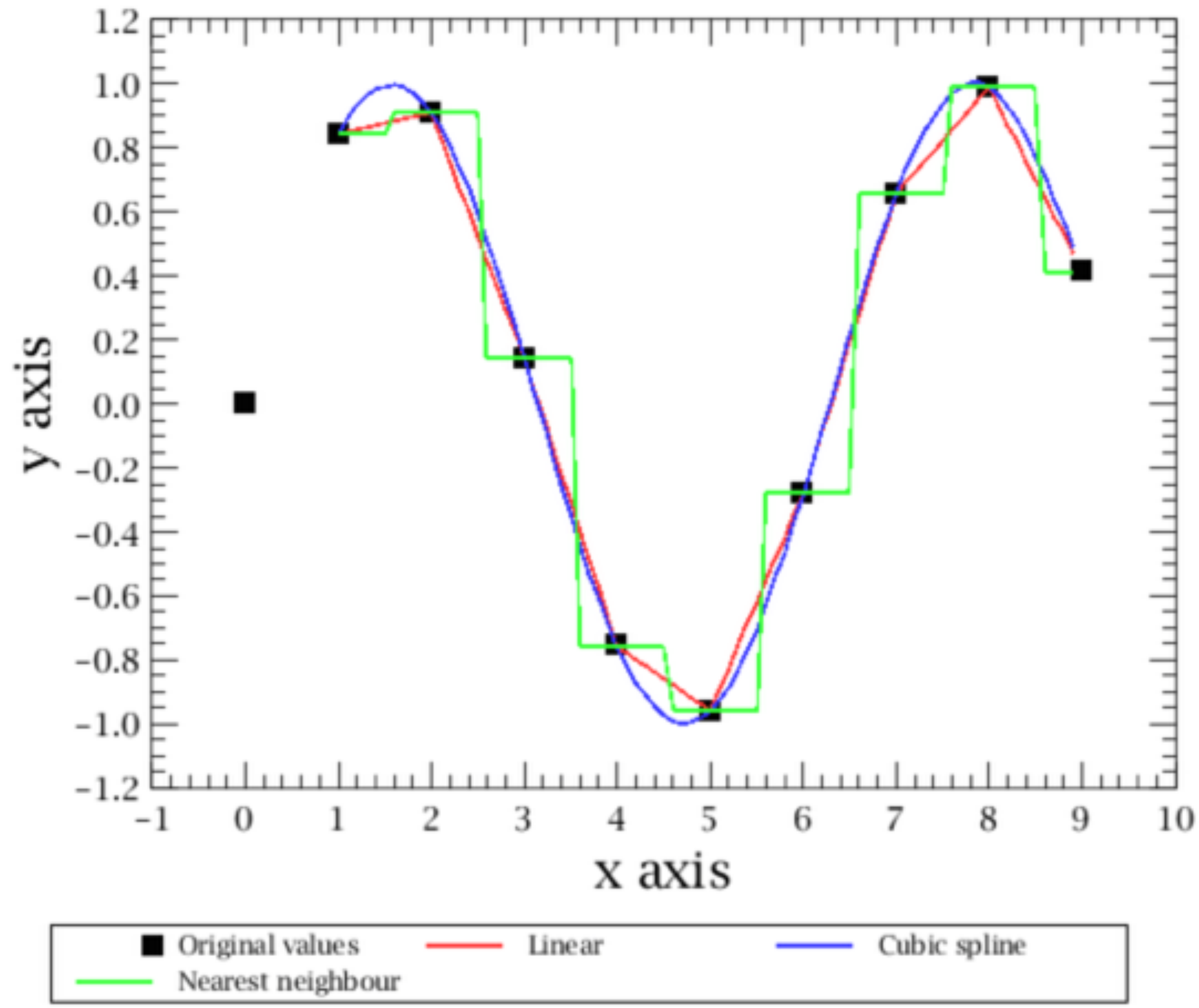
Background

- Topic: managing interest rate risk for linear (i.e. no options) fixed-income products
- Tools:
  - Yield Curve
  - Sensitivity: PV01 & IV01
  - Hedging = Product $\otimes$ Curve
- Note: our approach is fully analytic!
  - analytic vs numerical

Yield Curve

- Possibilities:
  - Parametric fitting (e.g. Nelson-Siegel-Svenson)
  - Interpolation
- Good features:
  - some level of smoothness
  - price back the market
  - continuous and positive forward rates
  - local construction method
  - local hedge
- Curve determines the risk space



Product

- Fundamental Asset Price Formula (F.A.P.F.):
  - the present value *(PV)* of any product is equal to the sum of its discounted cash flows
- PV is used to compute sensitivities w.r.t.
  - curve zero rates (PV01)
  - curve instrument rates (IV01)
- How are PV01 and IV01 related?
  - change of basis (Jacobian)

Hedging

- Use either PV01 or IV01 to determine hedging (replicating) positions
- Will the positions be the same?
  - Recall: curve determines the risk space
- Standard approach: perturbative, first order
  - Need to hedge frequently if product has large convexity

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Set Up

Definitions I

- Curve Instruments Set $\mathcal{I}$

$$\mathcal{I} = \{\text{cash}_{\text{2w}},\ \text{cash}_{\text{1m}},\ \text{cash}_{\text{3m}},\ \text{cash}_{\text{6m}},\ \text{swap}_{\text{1y}},\ \text{swap}_{\text{5y}},\ \text{swap}_{\text{10y}},\ \text{swap}_{\text{20y}}\}$$

- Curve Zero Nodes $\mathcal{N}$

$$\mathcal{N} = \{(t_1, r_1),\ (t_2, r_2),\ \ldots,\ (t_n, r_n)\}$$

- Bootstrap $\mathcal{B}$

$$\mathcal{B} : \mathcal{I} \longrightarrow \mathcal{N}$$

Definitions II

- Curve $\gamma \equiv \gamma_{\mathcal{N}}^{\text{int}} : \mathcal{N} \longrightarrow \mathbb{R}$
  - Interpolation dependent
- Discounts $D(t) = e^{-\int_0^t r(t')dt'}$ $D(t) = \left(\frac{1}{1+\frac{r}{n}}\right)^{-nt}$
  - Curve dependent
- Product's intermediate cash flows are discounted with corresponding discount factors

$$\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$$

- PV via F.A.P.F.

Definitions - Summary

$$\mathcal{I} \xrightarrow{\mathcal{B}} \mathcal{N} \xrightarrow{\gamma} \mathbb{R}^{\gamma} \xrightarrow{D} \mathcal{D} \xrightarrow{FAPF} \mathbb{R}^{PV}$$

Bump Curve Zero Rates & PV01

- For linear product (i.e. no optionality), when bumping curve zero rates, linear approximation is sufficient:

$$\delta PV(r) = \nabla PV \cdot \Delta r = \sum_i \frac{\partial PV}{\partial r_i} \delta r_i$$

- If only one node is bumped and the bump size is 1bp, then we find the standard PV01:

$$PV01_i = \frac{\partial PV}{\partial r_i} \delta r_i$$

Bump Instrument Rate & IV01

- For linear product (i.e. no optionality), when bumping instrument rates, linear approximation is sufficient:

$$IV01_i = PV(r(x_i + 1bp)) - PV(r(x_i))$$

- If only one instrument rate is bumped and the bump size is 1bp, then we *define* the IV01 as:

$$IV01_i = \frac{\partial PV}{\partial x_i} \delta x_i$$

Computing the Derivatives

- PV

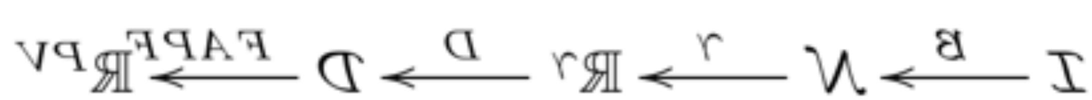
$$PV = \sum_{cf} A_{cf} D(t_{cf})$$

- PV01

$$\frac{\partial PV}{\partial r_i} = \sum_{cf} \frac{\partial PV}{\partial D_{cf}} \frac{\partial D_{cf}}{\partial r} \frac{\partial r}{\partial r_i}$$

- IV01

$$\frac{\partial PV}{\partial x_i} = \sum_{j} \frac{\partial PV}{\partial r_j} \frac{\partial r_j}{\partial x_i} = \sum_{cf} \frac{\partial PV}{\partial D_{cf}} \frac{\partial D_{cf}}{\partial r} \sum_{j} \frac{\partial r}{\partial r_j} \frac{\partial r_j}{\partial x_i}$$



Observations

- $\frac{\partial r_j}{\partial x_i}$ represents the bootstrap on the set $\mathcal{I}$;
- $\frac{\partial r}{\partial r_j}$ represents interpolation from the set $\mathcal{N}$;
- $\frac{\partial D_{cf}}{\partial r}$ depends on the choice of discounting convention;
- $\frac{\partial PV}{\partial D_{cf}}$ is related to the product only via its PV.

Note: the first 3 items are curve properties!

Only the last item is a product feature, through its intermediate cash flows.

Zero Rates from Cash and Swap Instruments

The Jacobian

- Jacobian is a curve property!

$$\mathcal{J} \equiv \mathcal{J}_x(r_1, r_2, \ldots, r_n) = \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(r_1, r_2, \ldots, r_n)} \qquad \Longleftrightarrow \qquad \mathcal{J}_{kl} = \frac{\partial x_k}{\partial r_l}$$

$$\mathcal{J}^{-1} = \frac{\partial(r_1, r_2, \ldots, r_n)}{\partial(x_1, x_2, \ldots, x_n)} \qquad \Longleftrightarrow \qquad (\mathcal{J}^{-1})_{lk} = \frac{\partial r_l}{\partial x_k}$$

- Both the Jacobian and its inverse are lower triangular matrices
    - Reasons: forward start and bootstrap

Cash

- Cash instruments are deposits: promise a pre-agreed (simply-compounded) interest over a pre-determined time on an initial invested amount
  - *s* is the forward start of cash instrument
  - $t = \tau + s$ is the location of the zero node in the curve
  - τ is the maturity of the deposit

$$D(s) = (1 + x \cdot \tau) D(t) \qquad x = \frac{1}{\tau}\left(\frac{D(s)}{D(t)} - 1\right)$$

- Note:
  - *D(s)* is typically associated with the first node, and is interpolation-dependent
  - *D(t)* is interpolation-independent, since *t* is the position of the node

Cash Example

- Consider 3 cash instruments (deposits):

$$D(s) = (1 + x_{2w} \cdot \tau_1)\, D(t_1)$$
$$D(s) = (1 + x_{1m} \cdot \tau_2)\, D(t_2)$$
$$D(s) = (1 + x_{3m} \cdot \tau_3)\, D(t_3)$$

- Without doing any calculation, we can already say that the Jacobian is lower triangular with some zero entries

$$\frac{\partial x_{2w}}{\partial r_1} \neq 0 \qquad \frac{\partial x_{2w}}{\partial r_2} = 0 \qquad \frac{\partial x_{2w}}{\partial r_3} = 0$$
$$\frac{\partial x_{1m}}{\partial r_1} \neq 0 \qquad \frac{\partial x_{1m}}{\partial r_2} \neq 0 \qquad \frac{\partial x_{1m}}{\partial r_3} = 0$$
$$\frac{\partial x_{3m}}{\partial r_1} \neq 0 \qquad \frac{\partial x_{3m}}{\partial r_2} = 0 \qquad \frac{\partial x_{3m}}{\partial r_3} \neq 0\,.$$

Swap

- (Interest rate) swap instruments are defined by the cashflows that are exchanged by the two parties.
- Argument similar to cash, but more complicated

$$D(s) - D(t) = x \sum_{k=1}^{N_{cf}} \alpha_k D(t_k) \qquad x = \frac{D(s) - D(t)}{\sum_{k=1}^{N_{cf}} \alpha_k D(t_k)}$$

Swap Example

- As for cash, consider 3 swap instruments

$$x_{1y} = \frac{D(s) - D(t_1)}{\sum_{k=1}^{N_{cf}} \alpha_k D(t_k)}$$

$$x_{3y} = \frac{D(s) - D(t_2)}{\sum_{k=1}^{N_{cf}} \alpha_k D(t_k)}$$

$$x_{5y} = \frac{D(s) - D(t_3)}{\sum_{k=1}^{N_{cf}} \alpha_k D(t_k)}$$

- Without doing any calculation, we can already say that the Jacobian is lower triangular with all the entries generically non-zero

$$\frac{\partial x_{1y}}{\partial r_1} \neq 0 \qquad \frac{\partial x_{1y}}{\partial r_2} = 0 \qquad \frac{\partial x_{1y}}{\partial r_3} = 0$$

$$\frac{\partial x_{3y}}{\partial r_1} \neq 0 \qquad \frac{\partial x_{3y}}{\partial r_2} \neq 0 \qquad \frac{\partial x_{3y}}{\partial r_3} = 0$$

$$\frac{\partial x_{5y}}{\partial r_1} \neq 0 \qquad \frac{\partial x_{5y}}{\partial r_2} \neq 0 \qquad \frac{\partial x_{5y}}{\partial r_3} \neq 0 \,.$$

Interpolation

Where needed?

- For curve construction
- In PV01 and IV01 calculations $\frac{\partial r(t)}{\partial r_i}$
- Curve property: defines the smoothness of the curve
- We can compute the derivative *exactly* for many interpolation methods:
  - linear
  - monotone-preserving cubic splines
  - Bessel-Hermite cubic spline
  - forward monotone convex spline (HW)

Hedging

Approach

- Purpose:
  - Replicate a portfolio such that fluctuations in the portfolio due to fluctuations in the underlying rates are balanced by fluctuations in the hedging instruments
  - The hedged portfolio is then immune to small changes in the yield curve
- Various methods:
  1. Fancier method: *waves* or *scenario* method
     - allows to separate risk of yield curve from instruments;
     - desirable when curve instruments are not the same as hedging instruments
  2. Standard method: bumping
     - our approach (we use the same set of instruments)

Some Notation

- $\mathcal{J}$ is the Jacobian introduced earlier
- Ψ will denote the matrix whose columns are the IV01 of the curve instruments $\mathcal{I}$
- Ψ_i, with $i = 1, \ldots, n$ will denote the columns of Ψ
- ψ will denote the IV01 of an arbitrary product that we want to hedge
- Π will denote the matrix whose columns are the PV01 of the curve instruments $\mathcal{I}$
- Π_i, with $i = 1, \ldots, n$ will denote the columns of Π
- ξ will denote the PV01 of an arbitrary product that we want to hedge
- ω will denote the vector with the hedging position

Sensitivities

- Recall:
  - PV01

$$PV01_i = \frac{\partial PV}{\partial r_i} \delta r_i$$

  - IV01

$$IV01_i = \frac{\partial PV}{\partial x_i} \delta x_i$$

  - related by the Jacobian

$$PV01 = \mathcal{J}^T \cdot IV01$$

IV01 representation

- Curve instruments IV01 matrix (upper triangular, due to bootstrap)

$$\Psi \equiv \Psi(\mathcal{I}) = \left(\begin{array}{c|c|c|c} \vdots & \vdots & \vdots & \vdots \\ \Psi_1 & \Psi_2 & \dots & \Psi_n \\ \vdots & \vdots & \vdots & \vdots \end{array} \right) = \left(\begin{array}{c|c|c|c} \psi_{11} & \psi_{12} & \dots & \psi_{1n} \\ 0 & \psi_{22} & \dots & \psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \psi_{nn} \end{array} \right)$$

- Product IV01 vector

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}, \qquad \psi_i \in \mathbb{R}$$

- Hedging: product's IV01 is a linear combination of the matrix column vectors

$$\psi_i = \sum_{j=1}^{n} \Psi_{ij} \omega_j \,, \qquad \omega_j \in \mathbb{R} \qquad \Longleftrightarrow \qquad \psi = \Psi \cdot \omega$$

- Solution for the positions: $\omega = \Psi^{-1} \cdot \psi$

PV01 in IV01 representation

- *Compute*: curve instruments PV01 matrix

$$\Pi = \mathcal{J}^T \cdot \Psi$$

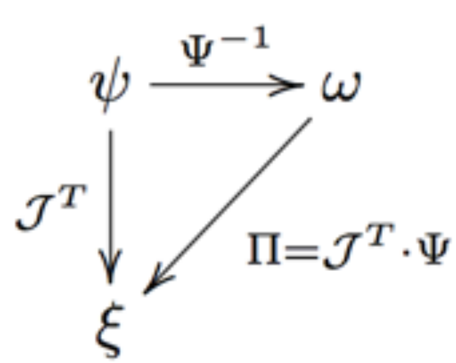
- Product PV01 vector

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}, \qquad \xi_i \in \mathbb{R}$$

- Then we can *derive* the relation

$$\xi = \Pi \cdot \omega$$

- Summarising diagram

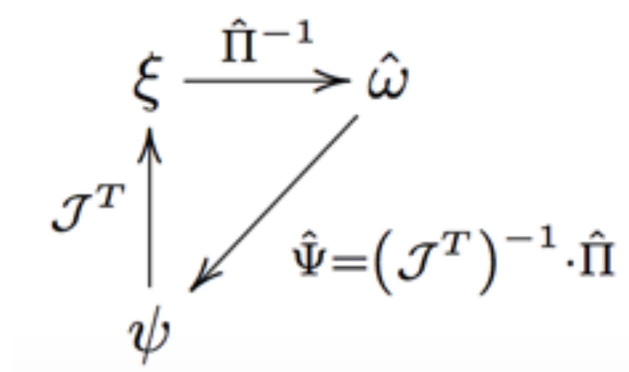


PV01 representation

- We can repeat the same procedure, but starting from the curve and product's PV01

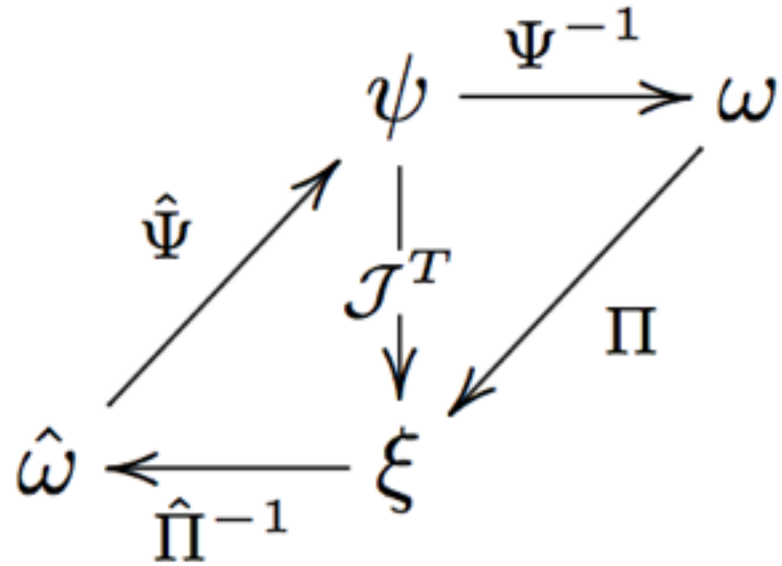
- We will find new positions $\hat{\omega} = \hat{\Pi}^{-1} \cdot \xi$

- Then we can *compute* the relation to the IV01, which will be given by the diagram



- And we can *derive* the IV01 for curve and product

Gluing the diagrams



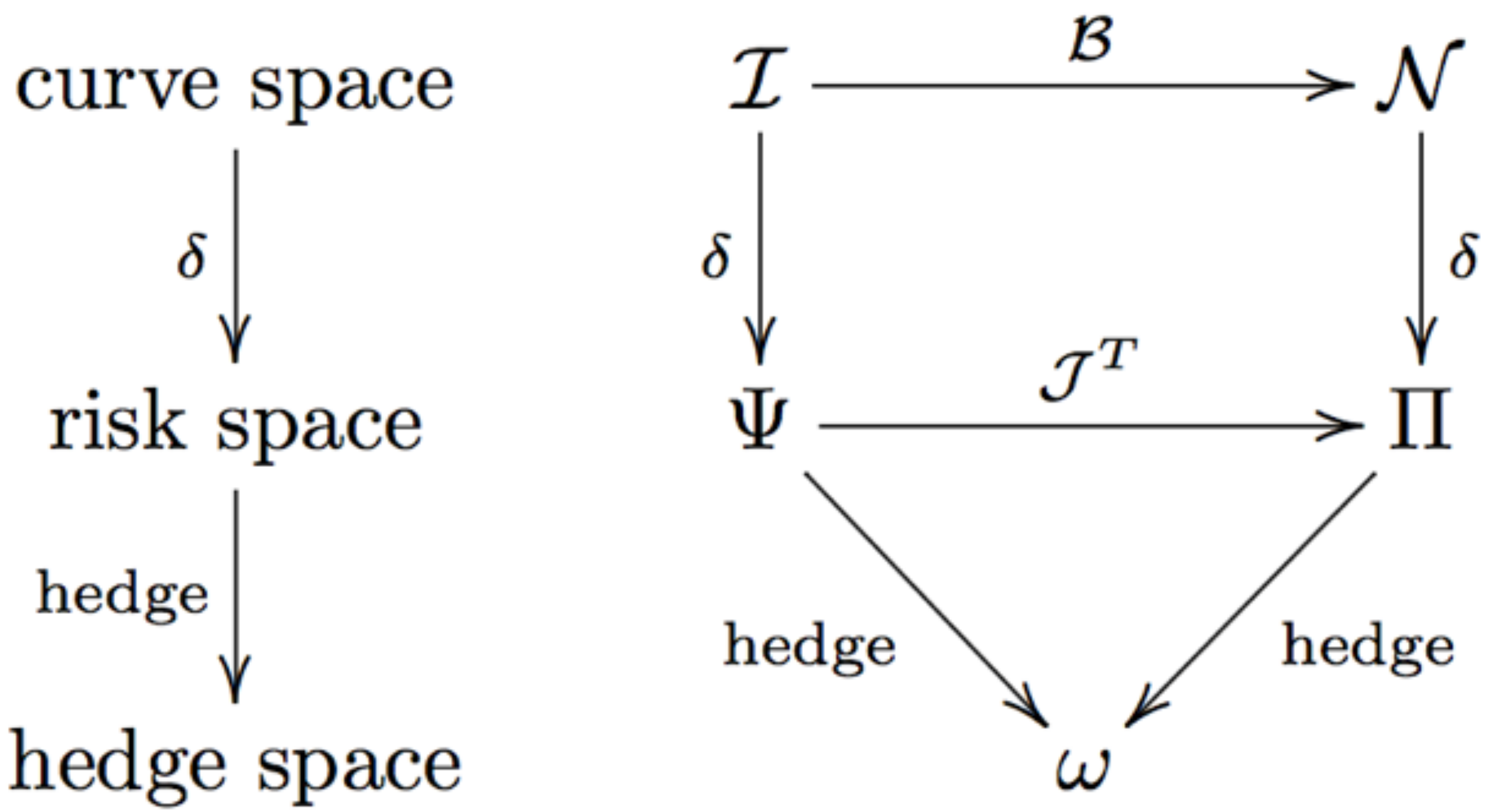
- But recall: PV01 and IV01 are not independent, but related by the Jacobian

$$\hat{\Pi} = \mathcal{J}^T \cdot \Psi$$

- This allows us to simplify and finally find:

$$\hat{\omega} = \omega \qquad \hat{\Pi} = \Pi \qquad \hat{\Psi} = \Psi$$

Hedging positions are strategy-invariant!



Risk spaces (range of the matrices) are the same!

$$\mathcal{R}_\Pi = \mathcal{R}_\Psi$$

Summary and Conclusions

Summary

- Bootstrapping prices back the market
- PV01 and IV01 are related by a change of basis
- Risk matrices span the whole risk space
- In this set up, hedging instruments and curve instruments are the same
- No numerical calculations

Possible future extensions

- From linear product to non-linear (options)
- From first order to higher order
- From single curve to multi curve
- Curve instruments are not the same as hedging instruments
- Hedging with waves

Thank you!

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